Adaptive Sliding-Mode Dynamic Control
For Path Tracking of Nonholonomic Wheeled Mobile Robot

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Abstract - This paper designs an adaptive sliding mode dynamic controller for the trajectory tracking of wheeled mobile robot. First, a kinematic controller based on backstepping technique is introduced to make the WMR follow a reference trajectory. Secondly, an adaptive sliding-mode dynamic controller (ASMDC) is proposed to make the actual velocity of the WMR reach the velocity command even in presence of uncertainties and disturbances. Using the Lyapunov stability theory, the stability analysis and convergence of the tracking errors are proved. Computer simulation results illustrate the effectiveness of the proposed controller.

Keywords: Adaptive control; Dynamic control; Kinematic control, Lyapunov method; Sliding mode control; Wheeled mobile robot; Trajectory tracking.

1. INTRODUCTION

Issues involving the control of nonholonomic wheeled mobile robots (WMRs) have been considered over recent decades. The system design of WMRs can be based on a kinematic or dynamic model. Many studies [1, 2, 3] are interested in kinematic tracking problems without taking into account the dynamics of WMRs. Nevertheless, considering only a kinematic model in a real trajectory problem is insufficient to reach a good tracking performance, since the error between the output of the velocity controller and the real velocity of WMRs. Thus, it more realistic to account for both kinematics and dynamics models in real trajectory tracking of WMRs. Several approaches have been investigated using back-stepping control [4, 5, 6], sliding mode control (SMC) [7, 8, 9], adaptive control [10, 11, 12], neural control [13, 14, 15] or fuzzy control [16, 17, 18] in order to overcome trajectory tracking problems. In [13], a control structure that makes possible the integration of a kinematic controller and a neural network (NN) computed-torque controller for mobile robots has been presented. The NN controller can deal with unmodeled bounded disturbances and/or unstructured dynamics of mobile robot. In [15], learning ability of neural networks has been used to design a robust adaptive backstepping controller that copes with disturbances and uncertainties. However, neural networks present many drawbacks such as slow convergence and difficulties in choosing the appropriate networks. Fuzzy logic controllers using different membership functions have been developed for mobile robot navigation [16, 20]. In [18], a tracking controller based on type 2 fuzzy logic theory has been presented to control mobile robot with unknown dynamics. The significant drawback of fuzzy logic controller is difficulties in determining fuzzy membership functions and fuzzy rules, especially for complicated systems. Classical SMC which offer
many great properties such as ability to deal with uncertainties, small tracking error and fast response, is a powerful control scheme for nonlinear systems [7, 8, 9]. In spite of claimed robustness properties, the resulting controller has a specific disadvantage. Its drawback is the chattering phenomenon, i.e. high frequency vibrations of the controlled system, which degrades the performance and may lead to instability. Therefore, the idea of integrating SMC with other control strategy (fuzzy control, neural networks, etc.) has been considered in [19-23] as a method to reduce the chattering phenomena and to enhance trajectory tracking.

Preserving the main advantages of the conventional SMC, an adaptive sliding mode dynamic controller (SMDC) is proposed to remove the chattering phenomenon and to treat both uncertainties and disturbances in the whole WMR system. The main contributions of this paper are (i) the velocity error between the kinematic velocity and actual velocity is adopted to build the sliding surface; (ii) the proposed controller compensates for the uncertainties and disturbances of WMR system; (iii) all the stability analysis and the convergence of the tracking errors to zeros are proven using the Lyapunov stability theory.

This paper is organized as follows: Section 2 presents the kinematic controller design. The complete equations of motion of ASMDC are described in section 3. Computer simulation results of SMDC and ASMDC are given in section 4. Finally, section 5 concludes this paper.

2. KINEMATIC CONTROLLER DESIGN

The nonholonomic WMR shown in Fig 1 is a vehicle with two driving wheels and a front passive wheel. The two wheels have the same radius denoted by \( r \) and are separated by \( 2b \). \( P_o \) defines the mass center of WMR.

![Nonholonomic wheeled mobile robot](image)

The kinematic model of the WMR under nonholonomic constraint of pure rolling and non-slippering is defined as follows:

\[
\dot{q} = \begin{bmatrix} \cos \varphi & 0 \\ \sin \varphi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} = S(q)v
\]  

(1)
where \( S(q) \in \mathbb{R}^{3 \times 2} \) and \( \nu \in \mathbb{R}^2 \) are the full rank velocity transformation matrix and velocity vector, respectively. \( \nu \) denotes the linear velocity of the WMR and \( \omega \) denotes the angular velocities of the WMR. The vector \( q(t) \in \mathbb{R}^3 \) is defined as:

\[
q = \begin{bmatrix} x_0 & y_0 & \phi \end{bmatrix}^T
\]

(2)

where \((x_0, y_0)\) are the actual position and linear velocity of \( P_0 \), \( \phi \) is the heading angle of the mobile robot.

The WMR has the nonholonomic constraint that the driving wheels purely roll and do not slip. This nonholonomic constraint \((m = 1)\) is written as the following:

\[
y_0 \cos \phi - \dot{x}_0 \sin \phi = 0
\]

(3)

The trajectory tracking problem is formulated by defining a reference mobile robot that gives a trajectory for the actual one to follow:

\[
\begin{align*}
\dot{x}_r &= v_r \cos \phi_r, \\
\dot{y}_r &= v_r \sin \phi_r, \\
\dot{\phi}_r &= w_r,
\end{align*}
\]

(4)

where \( q_r \) denotes the reference time varying position and orientation trajectory, and \( v_r(t) \) denotes the reference time varying linear velocity and \( w_r(t) \) denotes the reference time varying angular velocity.

The posture tracking error between the reference robot and the actual robot can be expressed as:

\[
q_e = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}^T = T\bar{q}
\]

(5)

where \( T = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \), \( \bar{q} = \begin{bmatrix} x_r - x_0 \\ y_r - y_0 \\ \phi_r - \phi \end{bmatrix} \).

The derivative of the posture tracking error given in (5) can be written as follows:

\[
\dot{q}_e = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} -v + we_2 + v_r \cos e_3 \\ -we_1 + v_r \sin e_3 \\ -w + w_r \end{bmatrix}
\]

(6)

The classic kinematic controller based on Backstepping method is designed to select the smooth velocity input [19]:

\[
v_c = \begin{bmatrix} v_c \\ w_c \end{bmatrix} = \begin{bmatrix} v_r \cos e_3 + k_1 e_1 \\ w_r + k_2 v_r e_2 + k_3 v_r \sin e_3 \end{bmatrix}
\]

(7)

where \( k_1, k_2 \) and \( k_3 \) are positive constants.
3. DYNAMIC CONTROLLER DESIGN

In this section we introduce two dynamic controllers, in order to guarantee that the motion of WMR can follow the desired velocity generated by the kinematic controller. First, we define the dynamic model of WMR. Second, we design a sliding-mode dynamic controller (SMDC) for the mobile robot. Finally, the ASMDC is formulated for solving the problem caused by system uncertainties and external disturbances. The stability proof of ASMDC is presented and confirmed by the Lyapunov stability theorem.

3.1 Dynamic model of WMR

According to the Euler-Lagrangian formulation, the nonholonomic wheeled mobile robot considering model errors and disturbances can be described as:

\[ M(q) \ddot{q} + V_m(q, \dot{q}) \dot{q} + F(\dot{q}) + G(q) + \tau_d = B(q) \tau - A^T(q) \lambda \]  

(8)

where \( M(q) \in R^{n \times n} \) is a symmetric positive definite inertia matrix of the system, \( V_m(q, \dot{q}) \in R^{n \times n} \), is a centripetal and Coriolis matrix, \( G(q) \in R^{n \times d} \) is the gravitational vector, \( F(\dot{q}) \in R^{n \times d} \) denotes the surface friction, \( \tau_d \in R^{n \times d} \) denotes bounded unknown disturbance including unstructured dynamic, \( B(q) \in R^{o \times (n-d)} \) is the input transformation matrix, \( \tau \in R^{(n-m) \times 1} \) is a control input vector, \( A \in R^{n \times n} \) is a matrix associated with nonholonomic constraints, \( \lambda \in R^{m \times 1} \) is the vector of Lagrange multipliers, and \( \dot{q} \) and \( \ddot{q} \) denote velocity and acceleration vectors respectively. Assuming that the mobile robot moves in the horizontal plane, in this case, \( G(q) = 0 \).

For control purpose, the constraint term \( A^T(q) \lambda \) in (8) should be removed by an appropriate transform. Substituting (1) and its differentiation for Eq. (8) and premultiplying by \( S^T(q) \), we obtain:

\[ \bar{M}(q) \dot{\nu} + \bar{V}_m(q, \dot{q}) \nu + \bar{F}(\dot{q}) + \bar{\tau}_d = \bar{B}(q) \tau \]  

(9)

where

\[ \bar{M} = S^T M S \in R^{2 \times 2}, \quad \bar{V}_m = S^T (M \dot{S} + V_m S) \in R^{2 \times 2}, \]
\[ \bar{F} = S^T F \in R^{2 \times 1}, \quad \bar{\tau}_d = S^T \tau_d \quad \text{and} \quad \bar{B} = S^T B. \]

Remark 1. In this dynamic model of the mobile robot, the passive self-adjusted supporting wheel influence is not taken into consideration, as it is a free wheel. This significantly reduces the complexity of the model controller design. However, the free wheel may be a source of substantial distortion, particularly in the changing case of its movement direction. This effect is reduced if we consider the small velocity of the robot.

Remark 2. Eq. (9) describes the behavior of the nonholonomic system in a new set of local coordinates, i.e. \( S \) is a Jacobian matrix that transforms velocities \( V \) in mobile base coordinates to velocities in Cartesian coordinates.

The effect of \( V_m \) can be removed from (9), in view of the distance between COM and the coordinate center of WMR is zero. The variables in (9) are defined as: \( \bar{M}(q) = \begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix} \) and
\[
\bar{B}(q) = \frac{1}{r} \begin{bmatrix}
1 & 1 \\
n & -n
\end{bmatrix}.
\]

By considering the surface friction and the disturbance torque as the external disturbances, we can reduce the dynamic equation (9) into the nominal dynamic model:

\[
\dot{\nu}(t) = E \cdot \tau(t),
\]

where the system matrix \( E \) is:

\[
E = \bar{M}^{-1}(q) \bar{B}(q) = \frac{1}{m.r.I} \begin{bmatrix}
I & I \\
n & -n
\end{bmatrix}.
\]

3.2 Sliding mode dynamic control

In this subsection, the SMC method is employed in designing a dynamic tracking controller which allows the actual velocities of WMR coincide with the control velocities generated from the kinematic controller. First, we design SMC for nominal system where disturbances are zero and the system parameters are known. In order to find the torque input, we propose the auxiliary velocity tracking error and its derivative as:

\[
e_c(t) = \begin{bmatrix} e_{c1} \\ e_{c2} \end{bmatrix} = \nu_c(t) - \nu(t)
\]

\[
\dot{e}_c(t) = \dot{\nu}_c(t) - \dot{\nu}(t)
\]

By selecting the PI-type sliding surface, the sliding surface is defined as:

\[
s(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} = e_c(t) + C \int e_c(t) dt
\]

where \( C \) is positive sliding surface integral constant. Yet, if the system is on the sliding surface \( s(t) = 0 \), \( e_c(t) = -C \int_0^t e_c(\tau) d\tau \) the tracking error \( e_c(\infty) \rightarrow 0 \) since \( C > 0 \).

The derivative of the sliding surface \( s(t) \) becomes:

\[
\dot{s}(t) = \dot{e}_c(t) + C e_c(t)
\]

From the concept of equivalent control law \( \tau_{eq} \) is stated by recognizing that \( \dot{s}(t) = 0 \) is a necessary condition for the state trajectory to stay in the sliding surface. Thus substituting (10) for (15), we obtain:

\[
\dot{s}(t) = (\dot{\nu}_c(t) - E \tau) + C e_c(t) = 0
\]

Therefore the equivalent control law \( \tau_{eq} \) is:

\[
\tau_{eq}(t) = E^{-1} \begin{bmatrix} \dot{\nu}_c(t) + C e_c(t) \end{bmatrix}
\]

where \( E^{-1} = \begin{bmatrix} -n & -n \\ -n & -n \end{bmatrix} \).
where \( K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \) and \( K_i \) is positive constant, and \( \text{sgn}(s) = [\text{sgn}(s_1) \text{ sgn}(s_2)]^T \).

The dynamic equation (10) in presence of uncertainties and disturbances becomes:

\[
\dot{v}(t) = E \tau + \tau_d(t) = \tilde{E} \tau(t) + \Delta E \tau(t) + \tau_d(t)
\]

(19)

where \( \tilde{E} \) is denoted as the nominal part of the system matrix \( E \) which is introduced by the parameters of WMR, \( m, r, I \) and \( b \). And \( \Delta E \) denoted the uncertainties of system matrix \( E \). \( \tau_d \) is the vector of external disturbances. Thus, we can introduce \( \delta(t) \) as the upper bound of uncertainties:

\[
\delta(t) = \begin{bmatrix} \delta_1(t) \\ \delta_2(t) \end{bmatrix} = \Delta E \tau(t) + \tau_d(t)
\]

(20)

Then the dynamic equation can be written as

\[
\dot{v}(t) = \tilde{E} \tau(t) + \delta(t),
\]

(21)

Hence, the sliding mode control law (18) can be written as:

\[
\tau(t) = \tau_{eq} + \tau_{sw}
= \tilde{E}^{-1} \left[ \dot{u}_c(t) + \beta \varepsilon_c(t) + K \text{sgn}(s) \right]
\]

(22)

where the switching gain \( K \) is chosen to compensate for the system uncertainties and disturbances.

In order to reduce the chattering phenomenon, we will use the arctangent function. Thus, replacing the \( \text{sgn} \) function by the arctangent function \( O(s, \varepsilon) \) in (22), implies:

\[
\tau(t) = \tau_{eq} + \tau_{sw}
= \tilde{E}^{-1} \left[ \dot{u}_c(t) + \beta \varepsilon_c(t) + K O(s, \varepsilon) \right]
\]

(23)

where \( O(s, \varepsilon) = \frac{2}{\pi} \arctan \left( \frac{s}{\varepsilon} \right), \varepsilon \) is a small positive constant.

### 3.3 Adaptive sliding mode dynamic control

Due to the difficulties encountered in measure of parameter variations of the WMR and the exact value of external disturbances in real applications, we propose an adaptive sliding-mode dynamic controller to estimate the upper bound of \( |\delta(t)| \). We suppose that the optimal bound of \( \delta^* \) and \( \gamma^* \) exist. The adaptive algorithm can be expressed as follows:

\[
\dot{\hat{\gamma}} = \begin{bmatrix} \dot{\hat{\gamma}}_1 \\ \dot{\hat{\gamma}}_2 \end{bmatrix} = \begin{bmatrix} \rho_1 s_1 O(s_1, \varepsilon_1) \\ 0 \end{bmatrix} \begin{bmatrix} \rho_2 s_2 O(s_2, \varepsilon_2) \\ 0 \end{bmatrix},
\]

(24)

where \( \hat{\gamma} \) is the estimated value of \( \gamma^* \) and \( \rho_i (i=1,2) \) is denoted as positive adaptation gain. The ASMDC for the WMR is designed as follows:
\[ \tau(t) = \tau_{eq} + \tau_{sw} \]
\[ = \tilde{E}^{-1} \left[ \dot{\hat{v}}_c(t) + \beta \dot{e}_c(t) + \hat{\gamma}(t) \sigma(s, \varepsilon) \right], \]  
(25)

And we define the estimation error as:
\[ \hat{\gamma} = \hat{\gamma}(t) - \gamma^*. \]  
(26)

The main goal is to choose an adaptive law to update the estimate \( \hat{\gamma}(t) \) such that \( s(t) \) converge to a zero vector.

The complete structure of ASMDC for mobile robot system is described by the following figure:

![Figure 2. Complete structure of ASMDC for mobile robot system](image)

**Theorem 1.** If the kinematic controller (7) and the ASMDC (25) are applied, both the posture tracking error \( q_e \) and the velocity tracking error \( e_c \) of WMR system with uncertainties and disturbances (21) will asymptotically converge to zero vectors.

**Proof.** Let the Lyapunov function candidate be defined as:
\[ L = L_1 + L_2 \]
(27)

where
\[ L_1 = \frac{1}{2} (\dot{e}_1^2 + \dot{e}_2^2) + \frac{1 - \cos e_3}{k_2} \]  
(28)
\[ L_2 = \frac{1}{2} s^T(t) s(t) + \frac{1}{2} \left( \frac{1}{\rho_1} \dot{\gamma}_1^2 + \frac{1}{\rho_2} \dot{\gamma}_2^2 \right) \]  
(29)

Substituting (6) and (7) for the time derivative of \( V_1 \) in (28), we obtain:
\[ \dot{L}_1 = -k_1 \dot{e}_1^2 - \frac{k_3 v_r \sin^2 e_3}{k_2} \leq 0. \]  
(30)

Therefore, if the reference velocity \( v_r \geq 0 \) then \( \dot{L}_1 \leq 0 \). Differentiating (29), we obtain...
\[
\dot{L}_2 = s^T(t)\dot{s} + \frac{1}{\rho_1} \tilde{\gamma}_1 \dot{\gamma}_1 + \frac{1}{\rho_2} \tilde{\gamma}_2 \dot{\gamma}_2
\]
\[
= s^T(t)\left[\dot{\gamma}_0(s, \varepsilon) - \delta(t)\right] + \frac{1}{\rho_1} \tilde{\gamma}_1 \dot{\gamma}_1 + \frac{1}{\rho_2} \tilde{\gamma}_2 \dot{\gamma}_2
\]
\[
= s^T(t)\left[-(\gamma^* + \tilde{\gamma}(t))o(s, \varepsilon) - \delta(t)\right]
\]
\[
+ \frac{1}{\rho_1} \tilde{\gamma}_1 \dot{\gamma}_1 + \frac{1}{\rho_2} \tilde{\gamma}_2 \dot{\gamma}_2
\]
\[
= s^T(t)\left[-\gamma^*o(s, \varepsilon) - \delta(t)\right]
\]
\[
+ \sum_{i=1}^{2} \tilde{\gamma}_i(t) \left[\frac{1}{\rho_i} \dot{\gamma}_i - s_i o(s, \varepsilon)\right]
\]
\[
(31)
\]

\(\dot{\gamma}\) is equal to \(\dot{\gamma}\) because \(\gamma^*\) is constant. Substituting (21) and (25) for (31), we obtain:
\[
\dot{L}_2 = s^T(t)\left[-\dot{\gamma}(s, \varepsilon) - \delta(t)\right] + \frac{1}{\rho_1} \tilde{\gamma}_1 \dot{\gamma}_1 + \frac{1}{\rho_2} \tilde{\gamma}_2 \dot{\gamma}_2
\]
\[
= s^T(t)\left[-(\gamma^* + \tilde{\gamma}(t))o(s, \varepsilon) - \delta(t)\right]
\]
\[
+ \frac{1}{\rho_1} \tilde{\gamma}_1 \dot{\gamma}_1 + \frac{1}{\rho_2} \tilde{\gamma}_2 \dot{\gamma}_2
\]
\[
= s^T(t)\left[-\gamma^*o(s, \varepsilon) - \delta(t)\right]
\]
\[
+ \sum_{i=1}^{2} \tilde{\gamma}_i(t) \left[\frac{1}{\rho_i} \dot{\gamma}_i - s_i o(s, \varepsilon)\right]
\]
\[
(32)
\]
From the adaptive law (26), one can get:
\[
\frac{1}{\rho_i} \dot{\gamma}_i - s_i o(s, \varepsilon) = 0, \forall t \geq 0
\]
\[
(33)
\]
Substituting (33) for (32)
\[
\dot{L}_2 = s^T(t)\left[-\gamma^*o(s, \varepsilon) - \delta(t)\right]
\]
\[
= \left[\gamma^*_1 s_1 o(s_1, \varepsilon) + \gamma^*_2 s_2 o(s_2, \varepsilon) + \delta_1(t) s_1 + \delta_2 s_2\right]
\]
\[
\leq \sum_{i=1}^{2} |\delta_i(t)| s_i o(s_i, \varepsilon) - \gamma^*_i s_i o(s_i, \varepsilon)
\]
\[
(34)
\]
From (30) and (34) we can conclude that \(\dot{L}\) is negative semi-definite. That is the posture tracking \(q_e\) and sliding surface \(s\) approach zeros vectors. It is noted in (14) that if \(s = 0\), then \(e_c = -\beta \int e_c(t) dt\) and it is clear that \(e_c(\infty) \to 0\). This completes the proof of theorem.

4. SIMULATION RESULTS

To demonstrate the performance and robustness of the proposed controller, in the presence of uncertainties and disturbances, extensive simulations for wheeled mobile robot shown in Fig. 1 are presented in this study.

Actual parameters of wheeled mobile robot are:
\( b = 0.15 \) m, \( r = 0.03 \) m, \( I = 3.75 \) kg m\(^2\), \( m = 4 \) kg. The reference trajectory as a function of time is selected as: \( x_r(t) = 0.2t + 0.3, \ y_r(t) = 0.5 + 0.25\sin(0.2\pi t) \).

It consists of a sinusoidal path for the wheeled mobile robot.

The variation of mass and inertia of WMR is described as follows:
\[
\begin{align*}
    m &: 4 \rightarrow 6kg \\
    I &: 3.75 \rightarrow 5kg \ m^2
\end{align*}
\]

The external disturbance imposed on the wheels of mobile robot is set at \( t > 10s \) as:
\[ \tau_d = \begin{bmatrix} 2\sin(3t) & 2\cos(3t) \end{bmatrix}^T \]
Figure 3. Tracking performance with SMDC under uncertainties and disturbances: (a) The trajectory tracking. (b) The tracking errors. (c) Input torques at the wheels. (d) Reference and actual velocities.
Figure 4. Tracking performance with ASMDC under uncertainties and disturbances: (a) The trajectory tracking. (b) The tracking errors. (c) Input torques at the wheels. (d) Reference and actual velocities. (e) The variation in adaptive law.
Figures 3 and 4 reveal that the effect of parameter variations are almost zeros for SMDC and ASMDC. These two controllers have excellent ability to remove the parametric uncertainties (mass and inertia of WMR). However, there is obviously a difference when external disturbances occurred. After 10s, the tracking ability of SMDC is highly degraded, but the tracking efficacy of ASMDC is still maintained. In addition the control torques shown in Figures 4.c is very smooth, if we compare it with the control torques obtained by SMDC (Fig 3.c). Moreover, Fig 4.d shows that actual linear and angular velocities of the ASMDC could keep up with desired ones even if external disturbances appeared, in contrast with actual linear and angular velocities of SMDC. Fig 4.e illustrates the adaptive laws (24) which tend toward some finite values. Thus, simulation result verifies the robustness of the proposed ASMDC in presence of external disturbances.

5. CONCLUSIONS

In this paper, we have presented two steps for the tracking controller that is kinematic controller and ASMDC for WMR with changing parameters and external disturbances. The kinematic controller is first introduced to provide velocity controller for the kinematic model. Then ASMDC is designed to make actual velocity of the mobile robot system reach the desired velocity command obtained by the kinematic controller. Thus, the proposed controller allows to WMR to track desired path successfully. The Lyapunov stability theory has been utilized to verify the convergence of both the posture tracking error and velocity tracking error. Simulation results have demonstrated that the ASMDC scheme gives the best performance in comparison with SMDC as system uncertainties and external disturbances appeared.

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